* UNIT - II *

* Lebesgue Integral *

sec - Riemann Integral

→ **Riemann Integral:-** let $f : [a,b] \to R$ be a real valued function defined on the interval $[a,b]$ and let $p = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a,b]$ then each partition. We define the sums as follows.

$$U(P,f) = S = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) = s = \sum_{i=1}^{n} m_i \Delta x_i \text{ where } \Delta x_i = x_i - x_{i-1} \ \forall\ i = 1,2,\cdots n$$

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \text{ for } i = 1,2 \cdots n.$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \quad \text{for } i = 1,2,3 \cdots n$$

Now we define the upper Riemann-integral of 'f' with respect to 'x' over $[a,b]$ as follows.

$$R\overline{\int_a^b} f\, dx = \inf_P S(P,f)$$

Now we define the lower Riemann-integral of 'f' w.r.to 'x' over $[a,b]$ as follows.

$$R\underline{\int_a^b} f\, dx = \sup_P s(P,f)$$

where infimum and supremum taken overall partition 'p' of $[a,b]$

we say that 'f' is a Riemann integral on $[a,b]$ if

$$R\underline{\int_a^b} f\, dx = R\overline{\int_a^b} f\, dx = R\int_a^b f\, dx$$

i.e., 'f' is Riemann-Integral.

Note:-

let $p = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of

$[a,b]$ and 'f' be a bounded real valued function defined on $[a,b]$. we have $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$ and $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$

$\forall\ i = 1,2,3 \cdots n$ then we define the step function $\phi(x)$ and $\varphi(x)$ on $[a,b]$ as follows.

$\varphi(x) = M_i,\ x\in[x_{i-1},x_i],\ i=1,2\cdots n$

$\phi(x) = m_i,\ x\in[x_{i-1},x_i],\ i=1,2\cdots n$

Also $\int_a^b \varphi(x)\,dx = \sum_{i=1}^{n} M_i \Delta x_i = S = U(P,f)$ —①

$\int_a^b \phi(x)\,dx = \sum_{i=1}^{n} m_i \Delta x_i = s = L(P,f)$ —②

$\Rightarrow \phi(x) \le f(x) \le \varphi(x)\ \forall\ x\in[a,b]$

Also $R\underline{\int_a^b} f(x)\,dx = \sup s(P,f) = \sup_P \sum_{i=1}^{n} m_i \Delta x_i = \sup_P \int_a^b \phi(x)\,dx$

i.e., $R\underline{\int_a^b} f(x)\,dx = \sup_{\phi(x)\le f(x)} \int_a^b \phi(x)\,dx$

Similarly $R\overline{\int_a^b} f(x)\,dx = \inf_{\phi(x)\le f(x)} \int_a^b \varphi(x)\,dx$

Sec: The Lebsegue Integral of a bounded function over a set of Finite measure *

Characteristic function:- let 'x' be a non-empty set and $A \subseteq x$ (or) $A\subset X$. Then we define the characteristic function of 'A' (χ_A) as follows.

$\chi_A = 1$ if $x\in A$

$= 0$ if $x \notin A$

Note:-

1. we know that if $\mathbb{R}$ is real line and $A\subset\mathbb{R}$ then we have χ_A is measurable $\Leftrightarrow$ A is measurable.
2. $\chi_{A'} = 1-\chi_A$
3. $\chi_{A\cap B} = \chi_A\cdot\chi_B$
4. $\chi_{A\cup B} = \chi_A+\chi_B-\chi_{A\cap B}$

Simple Measurable Function:- A measurable function defined on a real line is called "simple function" if it range is finite. Let $\{a_1, a_2, a_3, \ldots a_n\}$ be the range of a simple function 'f' defined on a measurable set $E \subset R$ then $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ where $E_i = \{x / f(x) = a_i\} \forall i = 1, 2, 3 \ldots n$ where a_i's are distinct and non-zero. This representation of 'f' is called canonical form of simple function which is a linear combination of characteristic function.

Note:- $f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ vanishes outside. The set $\bigcup_{i=1}^{n} E_i$ of finite measurable.

Lebesgue Integral of a simple function:- let 'ϕ' be a simple function defined on 'R' also $\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ vanishes at outside the set $\bigcup_{i=1}^{n} E_i$ of finite measurable then we define the Lebesgue integral of a simple function as $\int \phi(x) dx = \sum_{i=1}^{n} a_i m(E_i)$ we denote as $\int \phi$ (or) $\int \phi(x) dx$ thus $\int \phi = \int \phi(x) dx = \sum_{i=1}^{n} a_i m(E_i)$.

Note:-

1. If 'E' is measurable, subset of R, then we have $\phi \chi_E$ is a simple function.

2. $\int \phi \chi_E = \sum_{i=1}^{n} a_i m(E_i \cap F)$ we denote the notation $\int \phi \chi_E$ or $\int_E \phi$

3. If 'f' is a Riemann integral function on $[a, b]$ then we have $f \in R$ on $[a, b]$.

4. $\sum_{i=1}^{n} \Delta x_i = b - a$

5. If $\phi \geq 0$ almost everywhere Then $\int \phi \geq 0$

Problem:- let 'f' be a function defined on 'R' as follows
$f(x) = 0$ if 'x' is rational
$f(x) = 1$ if 'x' is irrational, Then prove that 'f' is not Riemann integral on $[a,b]$.

solution:- Given that 'f' be a function defined on R as follows $f(x)=0$ if x is rational
$f(x)=1$ if x is irrational.

let $P = \{a = x_0 < x_1 < x_2 \cdots < x_n = b\}$ be a partition on $[a,b]$

Now $M_i = \sup f(x)$
$= x \in [x_{i-1}, x_i]$
$= \sup[0,1]$
$= 1.$

$m_i = \inf f(x)$
$= x \in [x_{i-1}, x_i]$
$= \inf[0,1]$
$= 0$

$\therefore m_i = 0, M_i = 1 \forall i$

Now $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$
$= \sum_{i=1}^{n} \Delta x_i$
$= b-a$

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$
$= 0$

$\therefore R\overline{\int_a^b} f(x)\,dx = \inf_P U(P,f) = \inf(b-a) = b-a$ —①

$R\underline{\int_a^b} f(x)\,dx = \sup_P L(P,f) = \sup(0) = 0$ —②

$\therefore$ ① $\neq$ ②

i.e., $R\overline{\int_a^b} f(x)\,dx \neq R\underline{\int_a^b} f(x)\,dx$

$\therefore$ 'f' is not Riemann-integral

Lemma ①:- let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ with $E_i \cap E_j = \phi$ for $i \neq j$ where E_i's are pairwise disjoint and a_i's are all distinct non-zero values taken by 'ϕ'. Suppose each E_i is measurable set of finite measurable then $\int \phi \, dx = \sum_{i=1}^{n} a_i m(E_i)$.

Proof:- Given that $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$

$\Rightarrow \phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ is a canonical form of simple function 'ϕ' where $a_1, a_2, a_3, \dots a_n$ are distinct non-zero values taken by ϕ and $E_i = \{x \mid \phi(x) = a_i\}$; $1 \leq i \leq n$

Now let $\phi(x) = \sum_{j=1}^{m} b_j \chi_{B_j}(x)$ where it is also $b_1, b_2, b_3 \dots b_m$ are distinct non-zero values taken by 'ϕ'.

$B_j = \{x \mid \phi(x) = b_j\}$; $1 \leq j \leq m$

But we know that $\phi(x) = \sum_{j=1}^{m} b_j \chi_{B_j}(x)$ vanishes outside the set of finite measurable.

so, by definition of lebsegue integral of a simple measurable function, we have $\int \phi(x) \, dx = \int \phi \, dx = \int \sum_{j=1}^{m} b_j \chi_{B_j}$

$$= \sum_{j=1}^{m} b_j m(B_j) \text{ — ①}$$

But we have to prove that $\int \phi = \sum_{i=1}^{m} a_i m(E_i) \ni a_1, a_2, a_3, \dots a_n = b_1, b_2, b_3, \dots b_m$ for $1 \leq i \leq n$, $1 \leq j \leq m$

we have $B_j = \bigcup_{a_i = b_j} E_i$, $1 \leq i \leq n$, $1 \leq j \leq m$ — ②

substitute eqn ② in eqn ①

$$\int \phi = \sum_{j=1}^{m} b_j m\left[\bigcup_{a_i = b_j} E_i\right]$$

$$= \sum_{j=1}^{m} b_j \sum_{a_i = b_j} m(E_i)$$

$= \sum_{a_i=b_1} b_1 m(E_i) + \sum_{a_i=b_2} b_2 m(E_i) + \cdots + \sum_{a_i=b_m} b_m m(E_i)$

$= \sum_{a_1=b_1} b_1 m(E_1) + \sum_{a_2=b_2} b_2 m(E_2) + \cdots + \sum_{a_n=b_m} b_m m(E_n)$

we have $i = 1, 2, 3 \ldots n$

$$\int \phi = \sum_{i=1}^{n} a_i m(E_i)$$

Hence proved.

Imp Proposition ②:- let 'ϕ' and 'φ' are two simple functions which vanishes outside a set of finite measurable then $\int (a\phi + b\varphi) = a\int\phi + b\int\varphi$ and if $\phi \geq \varphi$ almost everywhere then $\int\phi \geq \int\varphi$

Proof:- Given that 'ϕ' and 'φ' are two simple functions.

$\Rightarrow a\phi, b\varphi$ are also simple functions

$\Rightarrow a\phi + b\varphi$ is also simple function.

since 'ϕ' and 'φ' are simple functions, we have

$$\left.\begin{array}{l} \phi = \sum_{i=1}^{m} a_i \chi_{A_i} \\ \varphi = \sum_{j=1}^{n} b_j \chi_{B_j} \text{ where } A_i = \{x \mid \phi(x) = a_i\} \\ \qquad\qquad\qquad\qquad B_j = \{x \mid \varphi(x) = b_j\} \end{array}\right\} \text{---} ①$$

Also A_i's are pair-wise disjoint and $\bigcup_{i=1}^{m} A_i = R$

a_i's are distinct non-zero values taken by ϕ

similarly B_j's are pair wise disjoint and $\bigcup_{j=1}^{n} B_j = R$,

b_j's are distinct non-zero values taken by φ

Now the canonical representations of $a\phi, b\varphi$ are given as $a\phi = a\left[\sum_{i=1}^{m} a_i \chi_{A_i}\right]$, $b\varphi = b\left[\sum_{j=1}^{n} b_j \chi_{B_j}\right]$

$= \sum_{i=1}^{m} a a_i \chi_{A_i}$ $\qquad = \sum_{j=1}^{n} b b_j \chi_{B_j}$

$$\therefore a\phi + b\varphi = \sum_{i=1}^{m} a a_i \chi_{A_i} + \sum_{j=1}^{n} b b_j \chi_{B_j}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (a a_i + b b_j) \chi_{A_i \cap B_j}$$

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[since $\sum_{i=1}^{m}\sum_{j=1}^{n}(aa_i+bb_j)\chi_{A_i\cap B_j} = \sum_{i=1}^{m}\sum_{j=1}^{n} aa_i\chi_{A_i\cap B_j} +$

$\sum_{i=1}^{m}\sum_{j=1}^{n} bb_j\chi_{A_i\cap B_j}$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} aa_i\chi_{A_i}(x)\chi_{B_j}(x) + \sum_{i=1}^{m}\sum_{j=1}^{n} bb_j\chi_{A_i}(x)\chi_{B_j}(x)$

$= \sum_{i=1}^{m} aa_i\chi_{A_i}(x)\sum_{j=1}^{n}\chi_{B_j}(x) + \sum_{i=1}^{m}\chi_{A_i}(x)\sum_{j=1}^{n} bb_j\chi_{B_j}(x)$

$= \sum_{i=1}^{m} aa_i\chi_{A_i}(x)\left[\chi_{B_1}(x)+\chi_{B_2}(x)+\cdots+\chi_{B_j}(x)+\cdots+\right.$
$\left.\chi_{B_n}(x)\right] + \sum_{j=1}^{n} bb_j\chi_{B_j}(x)\left[\chi_{A_1}(x)+\chi_{A_2}(x)+\cdots+\chi_{A_i}(x)+\right.$
$\left.\cdots+\chi_{A_n}(x)\right]$

$= \sum_{i=1}^{m} aa_i\chi_{A_i}(x)[0+0+\cdots+1+\cdots 0] + \sum_{j=1}^{n} bb_j\chi_{B_j}(x)[0+0+$
$\cdots+1+\cdots 0]$

$= \sum_{i=1}^{n} aa_i\chi_{A_i}(x) + \sum_{j=1}^{n} bb_j\chi_{B_j}(x)$

i.e., $\sum_{i=1}^{m} aa_i\chi_{A_i}(x) + \sum_{j=1}^{n} bb_j\chi_{B_j}(x) = \sum_{i=1}^{m}\sum_{j=1}^{n}(aa_i+bb_j)\chi_{A_i\cap B_j}(x)$

$\Rightarrow a\sum_{i=1}^{m} a_i\chi_{A_i}(x) + b\sum_{j=1}^{n} b_j\chi_{B_j}(x) = \sum_{i=1}^{m}\sum_{j=1}^{n}(aa_i+bb_j)\chi_{A_i\cap B_j}(x)$

$\Rightarrow a\phi+b\psi = \sum_{i=1}^{m}\sum_{j=1}^{n}(aa_i+bb_j)\chi_{A_i\cap B_j}(x)$]

$\Rightarrow \int(a\phi+b\psi) = \int\left[\sum_{i=1}^{m}\sum_{j=1}^{n}(aa_i+bb_j)\chi_{A_i\cap B_j}(x)\right]$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} aa_i\, m(A_i\cap B_j) + \sum_{i=1}^{m}\sum_{j=1}^{n} bb_j\, m(A_i\cap B_j)$

$= a\sum_{i=1}^{m} a_i\sum_{j=1}^{n} m(A_i\cap B_j) + b\sum_{j=1}^{n} b_j\sum_{i=1}^{m} m(A_i\cap B_j)$

$= a\sum_{i=1}^{m} a_i\, m\left[\left(\bigcup_{j=1}^{n} A_i\right)\cap\left(\bigcup_{j=1}^{n} B_j\right)\right] + b\sum_{j=1}^{n} b_j\, m$
$\left[\left(\bigcup_{i=1}^{m} A_i\right)\cap\left(\bigcup_{i=1}^{m} B_j\right)\right]$

$= a\sum_{i=1}^{m} a_i\, m\left[A_i\cap\left(\bigcup_{j=1}^{n} B_j\right)\right] + b\sum_{j=1}^{n} b_j\, m\left[\left(\bigcup_{i=1}^{m} A_i\right)\cap B_j\right]$

$$= a\sum_{i=1}^{m} a_i\, m\left[A_i \cap \left(\bigcup_{j=1}^{n} B_j\right)\right] + b\sum_{j=1}^{n} b_j\, m\left[\left(\bigcup_{i=1}^{m} A_i\right)\cap B_j\right]$$

$$= a\sum_{i=1}^{m} a_i\, m(A_i \cap R) + b\sum_{j=1}^{n} b_j\, m(R\cap B_j)$$

$$= a\sum_{i=1}^{m} a_i\, m(A_i) + b\sum_{j=1}^{n} b_j\, m(B_j)$$

$$= a\int\phi + b\int\varphi$$

Thus, $\int[a\phi + b\varphi] = a\int\phi + b\int\varphi$

Now we prove that if $\phi \geq \varphi$ almost everywhere then $\int\phi \geq \int\varphi$. so,

consider $\phi \geq \varphi$ almost everywhere

$\rightarrow \phi - \varphi \geq 0$ almost everywhere

$\Rightarrow \int[\phi - \varphi] \geq 0$

$\Rightarrow \int\phi - \int\varphi \geq 0$

$\Rightarrow \int\phi \geq \int\varphi$

Hence proved.

AS ✓ imp **proposition 3:-** let 'f' be defined and bounded on an measurable set 'E' with $m(E) < \infty$. In order that

$$\inf_{f\leq\varphi} \int_E \varphi(x)\,dx = \sup_{f\geq\phi} \int_E \phi(x)\,dx \text{ for all simple functions}$$

'ϕ' and 'φ'.

It is necessary and sufficient that 'f' is measurable function.

(or)

State and prove necessary and sufficient condition that 'f' is measurable.

let 'f' be bounded measurable function defined on measurable set 'E' with $m(E) < \infty$ then 'f' is measurable $\Longleftrightarrow \inf_{f\leq\varphi} \int_E \varphi(x)\,dx = \sup_{f\geq\phi} \int_E \phi(x)\,dx$ where ϕ and φ are simple functions $\ni \phi \leq f \leq \varphi$.

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Proof:- Given that 'f' is a bounded function defined on a measurable set E with $m(E)<\infty$

suppose that 'f' is measurable function on E

Now we prove that $\inf_{f\le\varphi}\int_E \varphi(x)\,dx = \sup_{f\ge\phi}\int_E \phi(x)\,dx$ where 'ϕ' and 'φ' are simple functions ∋ $\phi\le f\le\varphi$

since we have 'f' is bounded on E

By definition, ∃ a constant $M>0$ ∋ $|f(x)|\le M\ \forall x\in E$

$\Rightarrow -M\le f(x)\le M\ \forall\ x\in E$

Now consider for each +ve integer 'n' and 'k' ∋ $|k|\le n$

i.e., $-n\le k\le n$ and let $E_k=\left\{x\in E \mid \frac{(k-1)M}{n}<f(x)<\frac{kM}{n}\right\}$ —①

since we have 'f' is measurable then eqⁿ ① is also measurable set

clearly E_k's are pairwise disjoint and $E=\bigcup_{k=-n}^{n}E_k$

$\Rightarrow m(E)=m\left[\bigcup_{k=-n}^{n}E_k\right]$

$\Rightarrow m(E)=\sum_{k=-n}^{n}m(E_k)$ —②

Now for each 'n' let us define simple functions $\varphi_n(x)$ and $\phi_n(x)$ on E as follows

$$\varphi_n(x)=\frac{M}{n}\sum_{k=-n}^{n}k\,\chi_{E_k}(x)\ \text{—③}$$

$$\phi_n(x)=\frac{M}{n}\sum_{k=-n}^{n}(k-1)\,\chi_{E_k}(x)\ \text{—④} \Rightarrow \phi_n(x)\le f(x)\le\varphi_n(x)$$

$\forall n$ and $\forall x\in E$

let us take 'ϕ' and 'φ' are sup and inf of $\phi_n(x)$ and $\varphi_n(x)$ respectively.

i.e, $\phi\ge\phi_n$ and $\varphi\le\varphi_n$

$\Rightarrow \int_E\phi\ge\int_E\phi_n$ and $\int_E\varphi\le\int_E\varphi_n$

$$\Rightarrow \sup_{\phi \le f} \int_E \phi \ge \int_E \phi_n \text{ and } \inf_{\varphi \ge f} \int_E \varphi \le \int_E \varphi_n$$

$$\Rightarrow \sup_{\phi \le f} \int_E \phi \ge \int_E \frac{M}{n} \sum_{k=-n}^{n} (k-1)\, \chi_{E_k}(x)$$

$$= \frac{M}{n} \sum_{k=-n}^{n} (k-1)\, m(E_k)$$

i.e., $$\sup_{\phi \le f} \int_E \phi \ge \frac{M}{n} \sum_{k=-n}^{n} (k-1)\, m(E_k)$$

$$-\sup_{\phi \le f} \int_E \phi \le -\frac{M}{n} \sum_{k=-n}^{n} (k-1)\, m(E_k) \quad \text{--- (5)}$$

Now $$\inf_{\varphi \ge f} \int_E \varphi \le \int_E \frac{M}{n} \sum_{k=-n}^{n} k\, \chi_{E_k}(x)$$

$$= \frac{M}{n} \sum_{k=-n}^{n} k\, m(E_k)$$

$$\inf_{\varphi \ge f} \int_E \varphi \le \frac{M}{n} \sum_{k=-n}^{n} k\, m(E_k) \quad \text{--- (6)}$$

since we have $\phi \le f \le \varphi$

$\Rightarrow \phi \le \varphi$

$\Rightarrow \varphi - \phi \ge 0$

$\Rightarrow \int_E (\varphi - \phi) \ge 0$

$\Rightarrow \int_E \varphi - \int_E \phi \ge 0$

$\Rightarrow 0 \le \int_E \varphi - \int_E \phi$

$$\Rightarrow 0 \le \inf_{\varphi \ge f} \int_E \varphi - \sup_{\phi \ge f} \int_E \phi$$

$$\le \sum_{k=-n}^{n} \frac{M}{n} k\, m(E_k) - \sum_{k=-n}^{n} \frac{(k-1)M}{n} m(E_k)$$

$$\le \frac{M}{n} \sum_{k=-n}^{n} (k-k+1)\, m(E_k)$$

$$= \frac{M}{n} \sum_{k=-n}^{n} m(E_k)$$

$= \frac{M}{n} m(E)$ [By eq^n (2)]

i.e., $0 \le \inf_{\varphi \ge f} \int_E \varphi - \sup_{\phi \le f} \int_E \phi \le \frac{M}{n} m(E)$

Now taking $\lim_{n \to \infty}$ we get $0 \le \inf_{\varphi \ge f} \int_E \varphi - \sup_{\phi \le f} \int_E \phi \le 0$

$$\Rightarrow \inf_{\varphi \ge f} \int_E \varphi = \sup_{\phi \le f} \int_E \phi$$

where 'ϕ' and 'φ' are simple functions $\ni \phi \le f \le \varphi$

Conversely suppose that $\inf_{\varphi \ge f} \int_E \varphi = \sup_{\phi \le f} \int_E \phi$ where 'ϕ' and 'φ' are simple functions $\ni \phi \le f \le \varphi$

we have to prove that 'f' is measurable function.

Now consider for each positive integer N $\exists$ a simple function $\varphi_n, \phi_n \ni \phi_n \le f \le \varphi_n \; \forall n$ and $\int_E \phi_n \ge k + \frac{M-1}{n}$, $\int_E \varphi_n \le k + \frac{M}{N}$ [By eq^n (1)]

$\Rightarrow -\int_E \phi_n \le -k - \frac{M}{n} + \frac{1}{n}$, $\int_E \varphi_n \le k + \frac{M}{n}$

Thus $\int_E \varphi_n - \int_E \phi_n \le \frac{1}{n} \; \forall n$ —— (4)

since we have ϕ_n and φ_n are simple functions

$\Rightarrow$ By definition, ϕ_n and φ_n are measurable functions

$\Rightarrow$ $\sup \phi_n$ and $\inf \varphi_n$ are measurable functions.

let $\sup \phi_n = \phi^*$ and $\inf \varphi_n = \varphi^*$

i.e., $\varphi^* \le \varphi_n$ and $\phi_n \ge \phi^*$ —— (A)

$\therefore$ ϕ^* and φ^* are measurable functions.

To prove 'f' is measurable:

It is enough to prove that $f^* = \phi^* = \varphi^*$ almost everywhere

since we have $\phi_n \le f \le \varphi_n \; \forall n$.

$\Rightarrow \sup \phi_n \le f \le \inf \varphi_n$

$\Rightarrow \phi^* \le f \le \varphi^*$ —— (8)

Now to show that $\phi^* = \varphi^* = f$ almost everywhere

It is enough to prove that $\phi^* = \varphi^*$ almost everywhere

For this let us consider, $A = \{x \mid \phi^*(x) \le \varphi^*(x)\}$ —— (9)

and $A = \bigcup_{\gamma > 0} A_\gamma$ —— (10)

where $A_\gamma = \{x \mid \phi^*(x) \le \varphi^*(x) - \frac{1}{\gamma}\}$ —— (11)

For this, consider $x \in A_\gamma$

$\Rightarrow \phi^*(x) < \varphi^*(x) - \frac{1}{\gamma}$ [$\because$ By (11)]

$\Rightarrow \phi_n(x) \le \phi^*(x) < \varphi^*(x) - \frac{1}{\gamma} \le \varphi_n(x) - 1/\gamma$ [by (A)]

$\Rightarrow \phi_n(x) \le \varphi_n(x) - \frac{1}{\gamma} \; \forall n.$

$$\Rightarrow \int_{A_\gamma} \phi_n(x) \le \int_{A_\gamma} \left[\varphi_n(x) - \frac{1}{\gamma}\right]$$

$$\Rightarrow \int_{A_\gamma} \left[\varphi_n - \frac{1}{\gamma}\right] \ge \int_{A_\gamma} \phi_n(x)$$

$$\Rightarrow \int_{A_\gamma} \varphi_n - \int_{A_\gamma} \phi_n \ge \int_{A_\gamma} \frac{1}{\gamma}$$

$$\ge \frac{1}{\gamma} \int_{A_\gamma} (1)$$

$$\ge \frac{1}{\gamma} \int \chi_{A_\gamma}(x)$$

i.e., $\frac{1}{\gamma} m(A_\gamma) \le \int_{A_\gamma} \varphi_n - \int_{A_\gamma} \phi_n$

$\le \frac{1}{n}$ [by (7)]

i.e., $\frac{1}{\gamma} m(A_\gamma) \le \frac{1}{n} \; \forall n$

Taking $\lim_{n \to \infty}$ we get $\frac{1}{\gamma} m(A_\gamma) \le 0$

$\Rightarrow m(A_\gamma) \le 0$

But we know that $m(A_\gamma) \ge 0$

Thus $m(A_\eta)=0$

eq^n (10) $\Rightarrow A=\cup A_\eta$

$m(A)=m(\cup A_\eta)$

$=\sum m(A_\eta)$

$=0$

i.e., $m(\{x \mid \phi^*(x) \le \varphi^*(x)=0\})$ — (12)

similarly, we can prove that $m(\{x \mid \phi^*(x) \ge \varphi^*(x)=0\})$ — (13)

From (12) and (13), we have

$m(\{x \mid \phi^*(x)=\varphi^*(x)=0\})$

$\Rightarrow \phi^*=\varphi^*$ almost everywhere

$\Rightarrow \phi^*=\varphi^*=f$ almost everywhere.

Hence 'f' is a measurable function

Hence proved.

Definition:- let 'f' be bounded measurable function defined on a measurable set 'E' of finite measure then we define the lebesgue integral of 'f' over E as follows.

$$\int_E f(x)\,dx = \inf_{\varphi \ge f} \int_E \varphi(x)\,dx = \sup_{\phi \le f} \int_E \phi(x)\,dx \text{ where}$$

infimum is taken $\forall$ simple function $\varphi \ge f$ and supremum is taken $\forall$ simple function $\phi \le f$ (or)

$\int_E f = I(f) = S(f)$ where $I(f) = \inf\{\int_E \varphi \mid \varphi$ is simple function, $\varphi \ge f\}$

$S(f) = \sup\{\int_E \phi \mid \phi$ is a simple function, $\phi \le f\}$

Note:-

1. Sometimes write $\int_E f(x)\,dx$ has $\int_E f$ we write $\int_a^b f(x)\,dx$ in place of $\int_{[a,b]} f(x)\,dx$

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2. If 'f' is a bounded measurable function that vanishes outside the set of finite measure then we write $\int f$ in place of $\int_E f$.

3. $\int_E f = \int f \chi_E$

Proposition (4):– let 'f' be a bounded function defined on $[a,b]$. If 'f' is riemann integral on $[a,b]$ then 'f' is measurable and $R\int_a^b f(x)dx = L\int_a^b f(x)dx$ (or) show that the right integral of 'f' over $[a,b]$ is same as the left integral of 'f' over $[a,b]$.

Proof:– suppose 'f' is a riemann integral on $[a,b]$

$\Leftrightarrow R\overline{\int_a^b} f(x)dx = R\underline{\int_a^b} f(x)dx.$

$\Leftrightarrow \inf\limits_{\varphi\geq f} \int_a^b \varphi(x)dx = \sup\limits_{\phi\leq f} \int_a^b \phi(x)dx$

where 'ϕ' and 'φ' are simple functions $\ni \phi \leq f \leq \varphi$

$\Leftrightarrow$ f is measurable function defined on $[a,b]$

[By th(3)]

since by data, we have 'f' is bounded function defined on $[a,b]$.

Here we have 'f' is a measurable function defined on $[a,b]$

Thus, 'f' is bounded measurable function defined on $[a,b]$

Then by definition, we have $\int_a^b f(x)dx = \inf\limits_{\varphi\geq f} \int_a^b \varphi \, dx = \sup\limits_{\phi\leq f} \int_a^b \phi \, dx$, where '$\phi$' and '$\varphi$' are simple functions $\ni$

$\phi \leq f \leq \varphi$

$\therefore$ we have $R\int_a^b f\,dx = L\int_a^b f\,dx = \int_a^b f\,dx$

i.e., $R\int_a^b f\,dx = L\int_a^b f\,dx$

i.e., $R\int_a^b f(x)\,dx = L\int_a^b f(x)\,dx$.

Hence proved.

Proposition (5):- If 'f' and 'g' are bounded measurable functions defined on a measurable set 'E' of finite measure then we have

Imp **

(a) $\int_E (af+bg) = a\int_E f + b\int_E g$

(b) if $f=g$ almost everywhere then $\int_E f = \int_E g$

(c) if $f \le g$ almost everywhere then $\int_E f \le \int_E g$ and hence $\left|\int_E f\right| \le \int_E |f|$

(d) if $A < f(x) < B$ then $A\cdot m(E) \le \int_E f \le B\cdot m(E)$

(e) If 'A' and 'B' are disjoint measurable set of finite measure $\int_{A\cup B} f = \int_A f + \int_B f$.

Proof:- Given that 'f' and 'g' are bounded measurable functions defined on a measurable set 'E' of finite measure.

To prove (a):- To prove this two simple function φ_1 and φ_2 with $\varphi_1 \ge f$ and $\varphi_2 \ge g$

$\Rightarrow \varphi_1+\varphi_2$ is also a simple function with $\varphi_1+\varphi_2 \ge f+g$ —(1)

By data, we have 'f' and 'g' are bounded measurable functions defined on a set E

$\therefore$ (1) $\Rightarrow \int_E f+g \le \int_E (\varphi_1+\varphi_2)$

$$\le \int_E \varphi_1 + \int_E \varphi_2$$

$$\le \inf_{\varphi_1 \ge f} \int_E \varphi_1 + \inf_{\varphi_2 \ge g} \int_E \varphi_2$$

$$\therefore \int_E f+g \le \int_E f + \int_E g \quad \text{—} ②$$

let us consider two simple functions ϕ_1 and ϕ_2 with $\phi_1 \le f$, $\phi_2 \le g$

$\Rightarrow \phi_1 + \phi_2$ is also a simple function with $\phi_1 + \phi_2 \le f+g$ — ③

By data we have 'f' and 'g' are bounded measurable functions defined on a set 'E'.

$\Rightarrow f+g$ is also a bounded measurable function defined on a set E.

$$③ \Rightarrow \int_E (f+g) \ge \int_E (\phi_1 + \phi_2)$$

$$\ge \int_E \phi_1 + \int_E \phi_2$$

$$\ge \sup_{\phi_1 \le f} \int_E \phi_1 + \sup_{\phi_2 \le g} \int_E \phi_2$$

$$\therefore \int_E (f+g) \ge \int_E f + \int_E g \quad \text{—} ④$$

From ② and ④, $\int_E f+g = \int_E f + \int_E g$ — ⑤

Now we have to show that $\int_E af = a\int_E f$ for any constant 'a'.

Case (i):- let $a > 0$

let 'φ' be a simple function $\ni \varphi \ge f \Rightarrow a\varphi \ge af$

since 'φ' is a simple function

i.e., $a\varphi$ is also simple function since 'f' is a bounded measurable function and 'a' is a constant

$\Rightarrow af$ is also bounded measurable function.

Now $\int_E af = \inf_{a\varphi \geq af} \int_E a\varphi$

$= a \inf_{\varphi \geq f} \int_E \varphi$

$= a \int_E f.$

Case (ii): let $a < 0$

let 'ϕ' be a simple function $\ni \phi \leq f \Rightarrow a\phi \leq af$

Now $\int_E af = \sup_{a\phi \leq af} \int_E a\phi$

$= a \sup_{\phi \leq f} \int_E \phi.$

$= a \int_E f$

Case (iii): clearly $\int_E af = a \int_E f$ —— ⑥

Hence for any constant 'a' $\int_E af = a \int_E f$

Hence for any constant 'b' $\int_E bg = b \int_E g$

Now consider $L.H.S = \int_E af + \int_E bg$

$= a \int_E f + b \int_E g$

$\therefore \int_E af + bg = a \int_E f + b \int_E g$

To prove (b):

since by data $f = g$ almost everywhere

$\Rightarrow f - g = 0$ almost everywhere —— ⑦

let 'φ' be a simple function with $\varphi \geq f - g = 0$ almost everywhere then $\int_E \varphi \geq 0 \Rightarrow \inf_{\varphi \geq f-g} \int_E \varphi \geq 0$ —— ⑧

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Now consider $\int_E f-g = \inf_{\varphi \geq f-g} \int_E \varphi \geq 0$ [By (B)]

i.e., $\int_E f-g \geq 0$

$$\int_E f - \int_E g \geq 0$$

$$\Rightarrow \int_E f \geq \int_E g \quad \text{—} (9)$$

Again, let 'φ' be a simple function with $\phi \leq f-g=0$ almost everywhere then $\int_E \phi \leq 0 \Rightarrow \sup_{\phi \leq f-g} \int_E \phi \leq 0$ — (10)

Now consider, $\int_E f-g = \sup_{\phi \leq f-g} \int_E \phi = 0$

$$= \int_E f - \int_E g \leq 0$$

$$= \int_E f \leq \int_E g \quad \text{—} (11)$$

From (9) and (11), we have $\int_E f = \int_E g$

To prove (C):- we have to prove that if $f \leq g$ almost everywhere then $\int_E f \leq \int_E g$ and $|\int_E f| \leq \int_E |f|$

since by data $f \leq g$ almost everywhere

$\Rightarrow (f-g) \leq 0$ almost everywhere

$\Rightarrow g-f \geq 0$ almost everywhere — (12)

let 'φ' be a simple function with $\varphi \geq g-f \geq 0$ almost everywhere then $\int_E \varphi \geq 0$ [By (A)]

$\Rightarrow \inf_{\varphi \geq g-f} \int_E \varphi \geq 0$ — (13)

Now consider, $\int_E g-f = \inf_{\varphi \geq g-f} \int_E \varphi$ [By definition of lebesgue integral of $g-f$ over E]

≥ 0 [By (B)]

i.e., $\int_E g-f \geq 0$

$$\int_E g - \int_E f \geq 0 \quad [\text{By case (a)}]$$

$$\Rightarrow \int_E g \geq \int_E f$$

$$\Rightarrow \int_E f \leq \int_E g$$

Now we have to prove that $\left|\int_E f\right| \leq \int_E |f|$

we know that $f \leq |f|$; $-f \leq |f|$

$$\Rightarrow \int_E f \leq \int_E |f| \ ;$$

$$\int_E -f \leq \int_E |f|$$

$$\Rightarrow \int_E f \leq \int_E |f| \ ; \ -\int_E f \leq \int_E |f|$$

$$\Rightarrow \left|\int_E f\right| \leq \int_E |f|$$

To prove (d):- we have to prove that if $A \leq f(x) \leq B$ then $A\,m(E) \leq \int_E f \leq B\,m(E)$

since by data, we have $A \leq f(x) \leq B$

$$\Rightarrow \int_E A \leq \int_E f(x) \leq \int_E B$$

$$\Rightarrow A \int_E 1 \leq \int_E f(x) \leq B \int_E 1.$$

$$\Rightarrow A \int_E \chi_E(x) \leq \int_E f \leq B \int_E \chi_E(x)$$

$$\Rightarrow A\,m(E) \leq \int_E f \leq B\,m(E)$$

To prove (e):- i.e., if 'A' and 'B' are two disjoint measurable sets of finite measure then $\int_{A\cup B} f = \int_A f + \int_B f$

since by data 'A' and 'B' are measurable sets of a finite measure

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$\Rightarrow A \cup B$ is also measurable sets of a finite measure.

since by data, A and B are disjoint

i.e., $A \cap B = \phi$

L.H.S $\int_{A\cup B} f = \int_{A\cup B} f \cdot 1$

$= \int f \chi_{A\cup B}(x)$

$= \int f \left[\chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)\right]$

$= \int f \left[\chi_A(x) + \chi_B(x)\right]$ [$\because$ we have $A \cap B = \phi$]

$= \int f \chi_A(x) + \int f \chi_B(x)$

$= \int_A f \cdot 1 + \int_B f \cdot 1$

$\therefore \int_{A\cup B} f = \int_A f + \int_B f$

Hence proved.

Imp *** Proposition (6) :- Bounded Convergence Theorem

imp Statement :- let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure and suppose that there is a real number $M \ni |f_n(x)| \leq M \ \forall\ x \in E$ and $n \in N$. If $f(x) = \lim_{n\to\infty} f_n(x) \ \forall\ x \in E$ then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

Proof :- By first version of Littlewood's Third principle i.e., 'E' be a measurable set of finite measure and $\{f_n\}$ be a sequence of measurable functions defined on E.

let 'f' be a measurable real valued function $\forall$ $x \in E$

we have $\lim_{n\to\infty} f_n(x) = f(x)$

Then given $\epsilon > 0$, $\delta > 0$ there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer $N \ni |f_n(x) - f(x)| < \epsilon$ $\forall\, x \notin A$

i.e., $x \in E - A$ $\forall\, n \geq N$

let $\epsilon' > 0$ and $\delta > 0$ be given by the above stated theorem $\exists$ an integer N and a measurable set $A \subset E$ with $m(A) < \delta \ni |f_n(x) - f(x)| < \epsilon'$ $\forall\, n \geq N$, $x \notin A$

Now consider $\epsilon' = \dfrac{\epsilon}{2m(E)}$, $\delta = \dfrac{\epsilon}{4M}$

For then ϵ' and δ we can find a measurable set $A \subset E$ with $m(A) < \dfrac{\epsilon}{4M}$ and an integer $N \ni$

$|f_n(x) - f(x)| < \dfrac{\epsilon}{2m(E)}$ $\forall\, n \geq N$ and $x \notin A$ — ①

Now consider, $\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right|$

$$\leq \int_A |f_n - f| + \int_{E-A} |f_n - f|$$

$$\leq 2M\, m(A) + \epsilon/2$$

[By data we have $|f_n(x)| \leq M$ $\forall\, n, x \in E$

$$\Rightarrow \lim_{n \to \infty} |f_n(x)| \leq \lim_{n \to \infty} M$$

$$\Rightarrow |f(x)| \leq M, \ \forall n \text{ and } x \in E$$

consider, $|f_n - f| \leq |f_n| + |f|$

$$\leq M + M$$

$$= 2M \text{ on } E$$

$$\Rightarrow \int_A |f_n - f| \leq \int_A 2M$$

$$= 2M \int_A (1)$$

$$= 2M \int \chi_A(x)$$

$$= 2M\, m(A)$$

i.e., $\int_A |f_n - f| \leq 2M\, m(A)$

Again consider, $|f_n - f| < \frac{\epsilon}{2m(E)}$ on $E-A$.

$$\int_{E-A} |f_n - f| < \int_{E-A} \frac{\epsilon}{2m(E)}$$

$$= \frac{\epsilon}{2m(E)} \int_{E-A} (1)$$

$$= \frac{\epsilon}{2m(E)} \int_{E-A} \chi_{E-A}(x)$$

$$= \frac{\epsilon}{2m(E)} m(E-A) \qquad \left[\because \frac{m(E-A)}{m(E)} = 1\right]$$

$$< \epsilon/_2 (1)$$

$$\therefore \int_{E-A} |f_n - f| < \epsilon/_2 \Big]$$

i.e., $\left|\int_E f_n - \int_E f\right| \leq 2M\, m(A) + \epsilon/_2$

$$< 2M \frac{\epsilon}{4M} + \epsilon/_2$$

$$< \epsilon/_2 + \epsilon/_2 < \epsilon$$

i.e., $\left|\int_E f_n - \int_E f\right| < \epsilon$

$$\lim_{n\to\infty} \left|\int_E f_n - \int_E f\right| \leq \lim_{n\to\infty} \epsilon$$

$$\left|\lim_{n\to\infty} \int_E f_n - \lim_{n\to\infty} \int_E f\right| \leq \lim_{n\to\infty} \epsilon$$

$$\lim_{n\to\infty} \int_E f_n - \int_E f \leq 0$$

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

$$\therefore \int_E f = \lim_{n\to\infty} \int_E f_n$$

Hence proved.

sec – The Integral of a Non-negative Function

Definition:– If 'f' is a non-negative function defined on a measurable set 'E', then we define the lebesgue integral of 'f' as $\int_E f = \sup\{\int_E h \mid h$ is a bounded bounded measurable function with $h \le f\}$ and $m\{x \mid h(x) \ne 0\} < \infty$ (or) finite.

imp

Proposition (8):– let 'f' and 'g' be two non-negative measurable functions then

(a). $\int_E cf = c\int_E f$, if $c > 0$

(b) $\int_E f+g = \int_E f + \int_E g$

(c) If $f \le g$ almost everywhere then $\int_E f \le \int_E g$

Proof:–

To prove (a):–

Given that 'f' is a non-negative measurable function defined on a measurable set E.

we have to show that if $c > 0$ then $\int_E cf = c\int_E f$

R.H.S $\Rightarrow c\int_E f = c \sup_{h \le f} \int_E h$, h is a bounded measurable function with $h \le f$ $m\{x \mid h(x) \ne 0\} < \infty$

$= \sup_{ch \le cf} \int_E ch$

$= \int_E cf$

$=$ L.H.S

$\therefore \int_E cf = c\int_E f$.

To prove (b):– Given that 'f' and 'g' are two non-negative measurable functions defined on a measurable set 'E'.

we have to show that $\int_E f+g = \int_E f + \int_E g$

let us consider two bounded measurable function 'h' and 'k' with $h \le f$ and $k \le g \Rightarrow m\{x \mid h(x) \ne 0\} < \infty$ and $m\{x \mid k(x) \ne 0\} < \infty$

we have $h \le f$ and $k \le g \Rightarrow h+k \le f+g$

$\therefore h+k$ is a bounded measurable function with $h+k \le f+g$

$\Rightarrow m\{x \mid (h+k)x \ne 0\} < \infty$

we have 'f' and 'g' are non-negative measurable functions defined on E.

$\Rightarrow f+g$ is also a non-negative measurable function defined on E.

Consider $\int_E f+g = \sup_{h+k \le f+g} \int_E h+k$

$$= \sup_{(h+k) \le f+g} \int_E h + \int_E k$$

$$= \sup_{h \le f} \int_E h + \sup_{k \le g} \int_E k$$

$$= \int_E f + \int_E g$$

To prove (c):- Given that $f \le g$

we have to prove that $\int_E f \le \int_E g$

By definition, we have $\int_E f = \sup_{h \le f} \int_E h$

h = bounded measurable function with $h \le f \Rightarrow m\{x \mid h(x) \ne 0\} < \infty$

put $\Delta_h = \{x \mid h(x) \ne 0\}$

then $m(\Delta_h) < \infty$ and

put $k = \min\{h, g\}$

$\Rightarrow k \le h, k \le g$

since we have 'h' is bounded and $k \le h$, we have 'k' is also bounded.

Also we have 'h' is measurable function and $k \le h$.

so, 'k' is also bounded measurable function.

$\Rightarrow$ 'g' is also bounded measurable function

let $x \notin \Delta_h \Rightarrow h(x) = 0$

$\Rightarrow k(x) = 0$

i.e., k vanishes when 'h' vanishes

$\Rightarrow \Delta_k \subset \Delta_h$

$\Rightarrow m(\Delta_k) \le m(\Delta_h)$

$< \infty$

$\therefore m(\Delta_k) < \infty$

since we have $k \le g$

$\Rightarrow \int_E k \le \int_E g$ — (A)

since we have $f \le g$ almost everywhere

$\Rightarrow h \le g$ almost everywhere

$\Rightarrow \exists$ a measurable set $B \ni h(x) \le g(x) \; \forall x \notin B$ and $m(B) = 0$

$\Rightarrow h(x) = g(x) \; \forall x \notin B$ — (1)

$\Rightarrow k(x) = g(x) \; \forall x \notin B$ — (2)

Thus by (1) and (2), we have

$k(x) = h(x) \; \forall x \notin B$

$\Rightarrow k = h$ almost everywhere

$\Rightarrow \int_E k = \int_E h$

$\Rightarrow \sup_{h \le f} \int_E h = \int_E k \le \int_E g$

$\Rightarrow \sup_{h \le f} \int_E h \le \int_E g$

$\Rightarrow \int_E f \le \int_E g$

Thus, if $f \le g$ almost everywhere then $\int_E f \le \int_E g$

Hence proved.

Note:-

1. A sequence $\{x_n\}$ of extended real valued functions defined on a set $E-A$ is said to be converges to a function 'f' pointwisely if $f_n(x) \to f(x)$ for exact $x \in E-A$.

2. $f_n \to f$ almost everywhere on 'E' if there is a measurable sets $A \subset S \ni m(A)=0$ and $f_n \to f$ pointwisely on $E-A$.

imp
** Lemma (9): Fatous Lemma

If $\{f_n\}$ is a sequence of non-negative measurable functions defined on a measurable set E and if $f_n(x) \to f(x)$ almost everywhere on E then $\int_E f \le \underline{\lim}_n \int_E f_n$.

Proof:- Suppose that $f_n \to f$ almost everywhere on 'E'. Then by definition $\exists$ a measurable set $A \subset E \ni$ $m(A)=0$ and $f_n(x) \to f(x)$ pointwise on $E-A$

Now we prove that $\int_E f \le \underline{\lim}_n \int_E f_n$

$\therefore$ we have $m(A)=0$ and 'f_n' is a sequence of non-negative measurable function $\int_A f_n = 0 \; \forall n$

$\Rightarrow \int_A f = 0$

since we know that $\int_E f = \int_A f + \int_{E-A} f$

$\Rightarrow \int_E f = 0 + \int_{E-A} f$

$\Rightarrow \int_E f = \int_{E-A} f$ —— ①

similarly, $\int_E f_n = \int_{E-A} f_n$ —— ②

To prove the required lemma it is enough to show that $\int_{E-A} f \le \underset{n}{\underline{\lim}} \int_{E-A} f_n$

For this let us consider 'h' is a bounded measurable function defined on $E-A \ni h \le f$ and 'h' vanishes outside of a set of finite measure

i.e., $m(\{x \mid h(x) \neq 0\}) < \infty$

put $\Delta_h = \{x \mid h(x) \neq 0\}$ —— ③

$\Rightarrow m(\Delta_h) < \infty$ —— ④

Now define $h_n(x) = \min\{h(x), f_n(x)\}$ —— ⑤

$\Rightarrow h_n \le h$ and $h_n \le f_n$ —— ⑥

Now if $x \notin \Delta_h$ then $h(x) = 0 \ni h_n(x) = 0 \; \forall n$ and $x \notin \Delta_{h_n}$

$h_n(x)$ vanishes when 'h' is vanishes

$\Rightarrow \Delta_{h_n} \subseteq \Delta_h$

$\Rightarrow m(\Delta_{h_n}) \le m(\Delta_h) < \infty$

i.e., $m(\Delta_{h_n}) < \infty$

since we have 'h' is bounded and $h_n \le h$, so, h_n is bounded.

since we have 'h' is a measurable function with $h_n \le h$ so h_n is measurable function

Thus, each h_n is bounded measurable function

$\Rightarrow \exists \; m > 0 \ni |h_n(x)| \le m \; \forall n, x$

$\therefore \{h_n\}$ is a sequence of bounded measurable defined on $E-A$. finite measure and $|h_n(x)| \le m \; \forall x, n$

since we have $f_n(x) \to f(x)$ pointwisely on $E-A$

i.e, $\lim_n f_n(x) = f(x) \; \forall x \in E-A$ —— (6)

$\lim_n h_n(x) = \lim_n \min\{h(x), f_n(x)\}$

$= \min\{h(x), \lim_n f_n(x)\}$

$= \min\{h(x), f(x)\}$

$= h(x)$

i.e, $\lim_n h_n(x) = h(x) \; \forall x \in E-A$

Hence $\{h_n\}$ is a sequence of bounded measurable functions defined on a measurable set $E-A$ of finite measure and $|h_n(x)| \leq m \; \forall x, n$

and $h(x) = \lim_n h_n(x) \; \forall x \in E-A$

Then by bounded convergence theorem, we have

$$\int_{E-A} h = \lim_n \int_{E-A} h_n$$

since we have h_n is a bounded measurable

$h_n \leq f_n \; \forall n$

$$\Rightarrow \int_{E-A} h = \lim_n \int_{E-A} h_n$$

$$\leq \lim_n \inf_{f_n \geq h_n} \int_{E-A} f_n$$

$$= \underline{\lim_n} \int_{E-A} f_n$$

i.e, $\int_{E-A} h \leq \underline{\lim_n} \int_{E-A} f_n$

$$\sup_{h \leq f} \int_{E-A} h \leq \underline{\lim_n} \int_{E-A} f_n$$

$$\Rightarrow \int_{E-A} f \leq \underline{\lim_n} \int_{E-A} f_n$$

Hence proved.

Theorem (10): Monotone Convergence Theorem

Imp **Statement:** let $\{f_n\}$ be an increasing sequence of non-negative measurable functions and let $f = \lim_n f_n$ almost everywhere then $\int f = \lim_n \int f_n$.

Proof: Given that $\{f_n\}$ be an increasing sequence of non-negative measurable functions and $f = \lim_n f_n$

claim: $\int f = \lim_n \int f_n$

Then by fatous lemma, we have $\int f \leq \underline{\lim}_n \int f_n$ —— (1)

we have to prove that $\int f \geq \overline{\lim}_n \int f_n$

since $\{f_n\}$ is an increasing sequence

i.e., $\{f_n\} = \{f_1, f_2, f_3, \ldots f_n, \ldots\}$

i.e., $f_1 \leq f_2 \leq f_3 \leq \ldots \leq f_n \leq \ldots$

i.e., $\{f_n\}$ is monotonically increasing sequence

[We know that a sequence $\{s_n\}$ of real numbers said to be monotonically increasing sequence if $s_n \leq s_{n+1}$ $(n = 1, 2, 3 \ldots)$ and monotonically decreasing sequence of $s_n \geq s_{n+1}$ $(n = 1, 2, 3, \ldots)$]

But we know that every monotonically increasing sequence is bounded. so, $\{f_n\}$ is bounded.

Then $f_n \leq f$ $\forall n$

$\Rightarrow \int f_n \leq \int f$

$\Rightarrow \lim_n \int f_n \leq \int \lim_n \int f$

$\Rightarrow \lim_n \sup_{f_n \leq f} \int f_n \leq \int f$

$\Rightarrow \overline{\lim}_n \int f_n \leq \int f$ —— (2)

From (1) and (2), we have $\overline{\lim}_n \int f_n \leq \int f \leq \underline{\lim}_n f_n$ —— (3)

But we know that $\overline{\lim}_n \int f_n \geq \underline{\lim}_n \int f_n$ —— (4)

from ③ and ④, we have

$$\int f = \overline{\lim_n} \int f_n = \underline{\lim_n} \int f_n$$

i.e., $\int f = \lim_n \int f_n$

i.e., $\int f = \lim_{n\to\infty} \int f_n$

Hence proved.

Corollary:- let $\{u_n\}$ be a sequence of non-negative measurable functions and let $f = \sum_{n=1}^{\infty} u_n$ then $\int f = \sum_{n=1}^{\infty} \int u_n$

Proof:- Given that $\{u_n\}$ be a sequence of non-negative measurable functions and $f = \sum_{n=1}^{\infty} u_n$ —①

claim:- $\int f = \sum_{n=1}^{\infty} \int u_n$

so let us consider $f_n = u_1+u_2+u_3+\cdots+u_n$

i.e., $f_n = \sum_{i=1}^{n} u_i$ —②

since here $u_1, u_2, u_3, \cdots u_n$ are all non-negative measurable functions.

② ⟹ $f_n \geq 0 \; \forall n$

each f_n is measurable function —Ⓐ

Again let us consider $f_{n+1} = u_1+u_2+\cdots+u_n+u_{n+1}$

i.e., $f_{n+1} = \sum_{i=1}^{n+1} u_i$

$$\Rightarrow f_{n+1} = \sum_{i=1}^{n} u_i + u_{n+1}$$

$$\Rightarrow f_{n+1} = f_n + u_{n+1} \quad \text{—③}$$

since here $u_1, u_2, u_3 \cdots u_n, u_{n+1}$ are non-negative measurable functions

i.e., $u_i \geq 0$ for $i = 1, 2, 3, \cdots n+1$

Take $i = n+1$ then $u_{n+1} \geq 0$

③ ⟹ $f_{n+1} - f_n = u_{n+1} \geq 0$

$\Rightarrow f_{n+1} - f_n \geq 0$

$\Rightarrow f_{n+1} \geq f_n$ —— (B)

Again let us consider $f_n = u_1+u_2+u_3+ \cdots + u_n + u_{n+1} \cdots$

i.e., $f_n = \sum_{n=1}^{\infty} u_n$ —— (4)

since here $u_1, u_2, u_3, \cdots u_n, u_{n+1} \cdots$ are all non-negative measurable functions —— (C)

By (1), we have, $f = \sum_{i=1}^{\infty} u_i$

$$\Rightarrow f = \lim_{n\to\infty} \sum_{i=1}^{n} u_i$$

$$f = \lim_{n\to\infty} f_n \text{ —— (D)}$$

Here by (A), (B), (C) and (D), $\{f_n\}$ is an increasing sequence of non-negative measurable function and $f = \lim_n f_n$

Then by monotone convergence theorem,

we have $\int f = \lim_n \int f_n$

$$\Rightarrow \int f = \lim_n \int \sum_{i=1}^{n} u_i$$

$$= \lim_n \sum_{i=1}^{n} \int u_i$$

$$\text{i.e., } \int f = \lim_{n\to\infty} \sum_{i=1}^{n} \int u_i$$

$$= \sum_{i=1}^{\infty} \int u_i$$

$$= \sum_{n=1}^{\infty} \int u_n$$

Hence proved.

Proposition (12):- let 'f' be a non-negative measurable function and $\{E_i\}$ is a disjoint sequence of measurable sets, let $E = \bigcup_i E_i$ then $\int_E f = \sum \int_{E_i} f$.

Proof:- let $u_i = f \chi_{E_i}$ —— (1)

Here 'u_i' is a non-negative measurable functions since E_i is a disjoint sequence of measurable sets we have χ_{E_i} is measurable and 'f' is a non-negative measurable function.

we have to prove that if $E = \bigcup_i E_i$ then $\int_E f = \sum \int_{E_i} f$.

For this, first we have to prove that $f = \sum_{i=1}^{\infty} u_i$

let $x \in E = \bigcup_i E_i$

$\Rightarrow x \in E_i$ for each 'i'

$$u_i(x) = (f\chi_{E_i})(x)$$
$$= f(x)\,\chi_{E_i}(x)$$
$$= f(x) \cdot 1$$
$$u_i(x) = f(x) \quad \text{---} (2)$$

Now $u_j(x) = f\chi_{E_j}(x)$

$$= f(x)\,\chi_{E_j}(x)$$
$$= f(x) \cdot 0$$
$$= 0$$
$$\therefore u_j(x) = 0$$

Now $R.H.S = \sum_{i=1}^{\infty} u_i(x)$

$$= \sum_{i=1}^{\infty} f(x)$$
$$= f(x)$$
$$\therefore f(x) = \sum_{i=1}^{\infty} u_i$$
$$\Rightarrow \int_E f = \sum_{i=1}^{\infty} \int_E u_i$$
$$= \sum_{i=1}^{\infty} \int_E f\,\chi_{E_i} \quad [\text{By } (1)]$$
$$= \sum_{i=1}^{\infty} \int f\,\chi_{E_i}\,\chi_E(x)$$
$$= \sum_{i=1}^{\infty} \int f\,\chi_{E_i \cap E}$$

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$= \sum_{i=1}^{\infty} f \chi_{E_i}$ [$\because E_i \cap E = E_i$]

$= \sum_{i=1}^{\infty} \int_{E_i} f$

$\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f.$

Definition:- A non-negative measurable function 'f' is said to be integrable over a set 'E'. If $\int_E f$ is finite

i.e., $\int_E f < \infty$

Proposition 13:- let f,g are two non-negative measurable functions. If 'f' is integrable over E and $g(x) \leq f(x)$ on E then 'g' is also integrable over E and $\int_E f-g = \int_E f - \int_E g.$

Proof:- Given that f,g are two non-negative measurable functions defined on a measurable set E. ~~If 'f' is integrable~~

Also given that $g(x) \leq f(x) \forall x \in E$

Assume that 'f' is an integrable over E.

By definition, $\int_E f < \infty$

we prove that 'g' is integrable over E

since we have $g(x) \leq f(x) \forall x \in E$

$\Rightarrow \int_E g(x) \leq \int_E f(x) < \infty$

$\Rightarrow \int_E g(x) < \infty$

i.e., 'g' is integrable over 'E'.

Now we prove that $\int_E f-g = \int_E f - \int_E g$

since we have $g \leq f$

$\Rightarrow g - f \leq 0$

$\Rightarrow f-g \geq 0$

And hence $f-g$ is a non-negative measurable function.

we know that $f = (f-g)+g$

$$\int_E f = \int_E (f-g)+g = \int_E f-g + \int_E g$$

$$\Rightarrow \int_E f - \int_E g = \int_E f-g$$

$$\therefore \int_E f-g = \int_E f - \int_E g$$

Hence proved.

Proposition 14:- let 'f' be a non-negative measurable function which is integrable over a measurable set 'E'. Then given $\epsilon > 0$ and $\delta > 0$ $\ni$ for every measurable set $A \subset E$ with $m(A) < \delta$ we have $\int_A f < \epsilon$

Proof:- Given that 'f' be a non-negative measurable function which is integrable over a measurable set E.

Then by definition $\int_E f = \sup\{h \mid h$ is a bounded measurable function with $h \leq f\}$ and

$m\{x \mid h(x) \neq 0\} < \infty$

Also we have $\int_E f < \infty$

claim:- $\int_A f < \epsilon$

Case(i):- let 'f' be bounded, then $\exists$ a real number $M \ni |f(x)| \leq M \;\forall x \in E$ —(1)

let $\epsilon > 0$ be given $\delta = \frac{\epsilon}{M}$ and 'A' be a measurable subset (of) 'E' with $m(A) < \delta$

L.H.S $\int_A f \leq |\int_A f|$

$\leq \int_A |f|$

$\leq \int_A M$ $\quad = \int_A M \cdot 1 = \int M \cdot \chi_A(x) = M \int \chi_A(x) = M\, m(A)$

$= M \cdot m(A)$

$\leq M\delta$

$\leq M\left(\frac{\epsilon}{M}\right)$

$< \epsilon$

$\therefore \int_A f < \epsilon$

case (ii):- let 'f' be unbounded $\forall n$

define $f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n \end{cases}$ ——— (2)

By (2) we have $|f_n(x)| \leq n$

so f_n is bounded $\forall n$

since we have 'f' is non-negative measurable function.

Each f_n is also non-negative measurable function

$\therefore \{f_n\} = \{f_1, f_2, f_3, \dots\}$

Here $f_1 \leq f_2 \leq \dots$

i.e., f_n is an increasing sequence

since we have $f \geq 0$ and $f_n \geq 0\ \forall n$ then $f - f_n \geq 0$

$\Rightarrow f_n \leq f\ \forall n$ and $f_n \to f$

Hence $\{f_n\}$ is an increasing sequence of non-negative measurable function defined on a measurable set E and $\lim_n f_n = f$ almost everywhere

Then by monotone convergence theorem

$$\int_E f = \lim_n \int_E f_n$$

let $\epsilon > 0$ be given

$\Rightarrow$ for $\epsilon/_2 > 0\ \exists N > 0 \ni \left|\int_E f - \int_E f_n\right| < \epsilon/_2\ \forall n \geq N$

for $n = N$ we have

$$\left|\int_E f - \int_E f_N\right| < \epsilon/2$$

$$\Rightarrow \left|\int_E f - f_N\right| < \epsilon/2$$

$$\Rightarrow \int |f - f_N| < \epsilon/2 \quad \text{--- (3)}$$

L.H.S $\int_A f = \int_A f - f_N + f_N$

$$= \int_A f - f_N + \int_A f_N$$

[from any def. $\int_A f_N \leq \epsilon$]

$$\leq \int f - f_N + \epsilon$$

$$< \epsilon/2 + \epsilon$$

$$< \epsilon$$

$$\int_A f < \epsilon$$

This is any case for $\epsilon > 0 \; \exists \; \delta > 0$ with $A \subset E$ and $m(A) < \delta$ we have $\int_A f < \epsilon$

Hence proved.

The General Lebesgue Integral

Definition:- The positive part of a function 'f' is defined as $f^+(x) = \max(f(x), 0)$

$$= (f \vee 0)(x)$$

The negative part of a function 'f' is defined as

$f^-(x) = \max(-f(x), 0) = (-f \vee 0)(x)$

Note:-

1. If 'f' is measurable function then both f^+ and f^- are measurable function.

2. $f = f^+ - f^-$

$|f| = f^+ + f^-$

Definition:- A measurable function 'f' is said to be integrable over a measurable set E. If both f^+ and f^- are integrable over E. In this case, we define

$\int_E f = \int_E f^+ - \int_E f^-$

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Proposition 15:- If 'f' and 'g' are integrable functions over a set E then

i, cf is integrable over E and $\int_E cf = c\int_E f$ for any constant 'c'

ii, f+g is integrable over E and $\int_E f+g = \int_E f + \int_E g$

iii, If $f \leq g$ almost everywhere $\int_E f \leq \int_E g$ also $|\int f| \leq \int |f|$

iv, If A and B are disjoint measurable sets then

$\int_{A\cup B} f = \int_A f + \int_B f$

Proof:- Given that 'f' and 'g' are integrable functions over a set 'E'.

Then by definition f^+, f^- and g^+, g^- are also integrable function over E.

i.e., $\left.\begin{array}{l}\int_E f^+ < \infty, \int_E f^- < \infty \text{ and} \\ \int_E g^+ < \infty, \int_E g^- < \infty\end{array}\right\}$ —①

To prove(i):-

i.e., cf is integrable over E and $\int_E cf = c\int_E f$

case(i):- let $c \geq 0$

consider, $(cf)^+(x) = \max((cf)x, 0)$

$= \max(c.f(x), 0)$

$= c \max\{f(x), 0\}$

$= cf^+$

$\Rightarrow (cf)^+ = cf^+$ for $c \geq 0$ —②

$\Rightarrow \int_E (cf)^+ = \int_E cf^+$

$= c\int_E f^+$

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$< \infty$ [$\because f$ is integrable] [By ①]

$\therefore \int_E (cf)^+ < \infty$

$\therefore (cf)^+$ is integrable over E if $c > 0$

Similarly,

$$(cf)^-(x) = \max(-(cf)x, 0)$$
$$= \max(-c \cdot f(x), 0)$$
$$= c \max(-f(x), 0)$$
$$= cf^-$$

$(cf)^- = cf^-$ for $c > 0$

$$\Rightarrow \int_E (cf)^- = \int_E cf^- = c \int_E f^- < \infty$$

$$\therefore \int_E (cf)^- < \infty$$

$\therefore (cf)^-$ is integrable over E if $c > 0$

Now $cf = (cf)^+ - (cf)^-$

$$\int_E cf = \int_E (cf)^+ - \int_E (cf)^-$$
$$= c\int_E f^+ - c\int_E f^-$$
$$= c\left[\int_E f^+ - \int_E f^-\right]$$
$$= c\int_E f$$

And also, we have $(cf)^+$ and $(cf)^-$ are integrable over E for $c > 0$. So, by definition we say that cf is integrable over E for $c > 0$

Case (iii):- let $c<0$

$\Rightarrow c=-d,\ d>0$

consider $(cf)^+(x) = \max(cf(x), 0)$

$= \max(c\cdot f(x), 0)$

$= \max(-d\cdot f(x), 0)$

$= d \max\{-f(x), 0\}$

$= d f^-$

$\Rightarrow (cf)^+(x) = d f^-(x)$

$$\int_E (cf)^+(x) = \int_E d f^-(x)$$

$$= d\int_E f^-(x)$$

$< \infty$

$\therefore (cf)^+$ is integrable over E.

Also $(cf)^+ = df^-$

$= -cf^-$ if $c<0$

similarly, $(cf)^- = -cf^+$ if $c<0$

since $(cf)^+, (cf)^-$ are integrable over E for $c<0$

we know that $cf = (cf)^+ - (cf)^-$

$$\int_E cf(x) = \int_E (cf)^+ - (cf)^-$$

$$= \int_E (cf)^+(x) - \int_E (cf)^-(x)$$

$$= \int_E -cf^- - \int_E -cf^+$$

$$= -c\int_E f^- + c\int_E f^+$$

$$= -c\left[\int_E f^- - \int_E f^+\right]$$

$$= c\left[\int_E f^+ - \int_E f^-\right]$$

$$= c\int_E f$$

$$\therefore \int_E (cf) = c\int_E f \text{ if } c<0$$

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Case (iii):- If $c=0$ then $c \cdot f = 0$

so $\int_E c f = 0 = 0 \int_E f$

$= c \int_E f$

i.e., $\int_E c f = c \int_E f$ if $c=0$

$\therefore$ For any constant c, we have $\int_E cf = c \int_E f$

To prove (ii):- i.e., $f+g$ is integrable over E and

$\int_E f+g = \int_E f + \int_E g$

consider $(f+g)^+(x) = \max(f+g(x), 0)$

$= \max(f(x)+g(x), 0)$

$= \max(f(x), 0) + \max(g(x), 0)$

$= f^+(x) + g^+(x)$

i.e., $(f+g)^+(x) = f^+(x) + g^+(x)$

$\Rightarrow (f+g)^+ = f^+ + g^+$

$\Rightarrow \int_E (f+g)^+ = \int_E (f^+ + g^+)$

$= \int_E f^+ + \int_E g^+$

$< \infty$

i.e., $\int_E (f+g)^+ < \infty$

$\Rightarrow (f+g)^+$ is integrable over E

similarly, we can prove that $(f+g)^- = f^- + g^-$

$\Rightarrow \int_E (f+g)^- = \int_E (f^- + g^-)$

$= \int_E f^- + \int_E g^-$

$< \infty$

$\therefore \int_E (f+g)^- < \infty$

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$\therefore (f+g)$ is integrable over E

Hence, we have $(f+g)^+$ and $(f+g)^-$ are integrable over E.

So, by definition, we say that $(f+g)$ is integrable over E

we know that $f+g = (f^+ + g^+) - (f^- + g^-)$

$$= (f^+ - f^-) + (g^+ - g^-)$$

$$\int_E f+g = \int_E (f^+ - f^-) + (g^+ - g^-)$$

$$= \int_E f^+ - f^- + \int_E g^+ - g^-$$

$$= \left(\int_E f^+ - \int_E f^-\right) + \left(\int_E g^+ - \int_E g^-\right)$$

$$\therefore \int_E f+g = \int_E f + \int_E g$$

To prove (iii):- i.e., if $f \leq g$ almost everywhere then $\int_E f \leq \int_E g$ Also $\left|\int f\right| \leq \int |f|$

Since we have $f \leq g$ almost everywhere

$$\Rightarrow g - f \geq 0 \text{ almost everywhere}$$

$$\Rightarrow \int_E g - f \geq 0$$

Now $\int_E g = \int_E (g-f) + f$

$$= \int_E g - f + \int_E f$$

$$\geq 0 + \int_E f$$

$$\text{i.e., } \int_E g \geq \int_E f$$

$$\Rightarrow \int_E f \leq \int_E g$$

we know that $f \leq |f|$ and $-f \leq |f|$

$$\Rightarrow \int f \leq \int |f| \text{ and } \int -f \leq \int |f|$$

$$\Rightarrow \int f \leq \int |f| \text{ and } -\int f \leq \int |f|$$

$\Rightarrow |\int f| \leq \int |f|.$

To prove (iv):- since we have 'A' and 'B' are disjoint measurable sets contained in E

i.e., $A \cap B = \phi$

$$L.H.S = \int_{A \cup B} f = \int_{A \cup B} f \cdot (1)$$

$$= \int f \chi_{A \cup B}(x)$$

$$= \int f \left[\chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)\right]$$

$$= \int f \left[\chi_A(x) + \chi_B(x)\right]$$

$$= \int f \chi_A(x) + \int f \chi_B(x)$$

$$= \int_A f \cdot (1) + \int_B f \cdot (1)$$

$$= \int_A f + \int_B f$$

$$\therefore \int_{A \cup B} f = \int_A f + \int_B f$$

Hence proved.

Proposition (16):- Lebesgue Convergence Theorem

Statement:- let 'g' be integrable over E and let $\{f_n\}$ be a sequence of measurable functions $\ni |f_n| \leq g$ $\forall n$ on E and they almost all x in E. we have $f(x) = \lim_{n \to \infty} f_n(x)$ then $\int_E f = \lim_n \int_E f_n$

Proof:- Given that $\{f_n\}$ is a sequence of measurable functions $\ni |f_n| \leq g$ $\forall n$ on E
And 'g' is integrable over E $\Big\}$ — (1)

So, each f_n is integrable over E.

Thus, $g + f_n$ is also integrable over E
Also, $g - f_n$ is also integrable over E $\Big\}$ — (A)

Also, given that for almost all 'x' in E we have $f(x) = \lim_{n} f_n(x)$ — (2)

Now (1) $\Rightarrow |f_n(x)| \leq g(x)$ $\forall$ 'n' on E

$\Rightarrow |\lim f_n(x)| \leq \lim g(x)$

$\Rightarrow |\lim f_n(x)| \leq g(x)$

$\Rightarrow |f(x)| \leq g(x)$ $\forall x$ — (3)

since we have 'g' is integrable over E (By (1)) → (4)

By (3) and (4), we have 'f' is integrable over E

Then $f+g$ is also integrable over E
$\Rightarrow f-g$ is also integrable over E } — (8)

since we have $|f_n| \leq g$ $\forall$ n on E

$\Rightarrow -g \leq f_n \leq g$ — (c)

$\Rightarrow g-g \leq f_n+g \leq g+g$

$\Rightarrow 0 \leq f_n+g \leq 2g$

Here $g+f_n \geq 0$ $\forall$ 'n' on E

i.e., $\{g+f_n\}$ is a sequence of non-negative measurable function and $g+f_n$ converges to $g+f$ almost everywhere on E.

$\int_E (g+f) \leq \underline{\lim} \int_E (g+f_n) = \underline{\lim} \int_E g + \underline{\lim} \int_E f_n$

i.e., $\int_E g + \int_E f \leq \int_E g + \underline{\lim} \int_E f_n$

$\Rightarrow \int_E f \leq \underline{\lim} \int_E f_n$ — (5)

since from (c), we have $(g-f_n) \geq 0$ $\forall$ 'n' on E

i.e., $\{g-f_n\}$ is a sequence of non-negative measurable functions on E and $g-f_n \to g-f$ almost everywhere on E.

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Then by fatous lemma, we have $\int_E g-f \leq \underline{\lim}_n \int_E g-f_n$

$$< \underline{\lim}_n \int_E g + \underline{\lim}_n \int_E (-f_n)$$

i.e., $\int_E g - \int_E f \leq \int_E g + \underline{\lim}_n \int_E (-f_n)$

$$\Rightarrow -\int_E f \leq \underline{\lim}_n \int_E -f_n$$

since we know that $\underline{\lim}_n (-x_n) = -\overline{\lim}_n (x_n)$

$$\Rightarrow -\int_E f \leq -\overline{\lim}_n \int_E f_n$$

$$\Rightarrow \int_E f \geq \overline{\lim}_n \int_E f_n \quad \text{--- (6)}$$

Then by (5) and (6), we have $\overline{\lim}_n \int_E f_n \leq \int_E f \leq \underline{\lim}_n \int_E f_n$ --- (7)

But we know that $\overline{\lim}_n \int_E f_n \geq \underline{\lim}_n \int_E f_n$ --- (8)

Then by (7) and (8), we have

$$\overline{\lim}_n \int_E f_n = \underline{\lim}_n \int_E f_n = \int f$$

i.e., $\lim_n \int_E f_n = \int_E f$

Hence $\int_E f = \lim_n \int_E f_n$

Hence proved.

Generalised Lebesgue Convergence Theorem

Statement:- let $\{g_n\}$ be a sequence of measurable functions which converge almost everywhere to an integrable function 'g' on E. let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g_n \; \forall n$ and $f_n \to f$ almost everywhere on E. If $\int_E g = \lim_n \int_E g_n$ then $\int_E f = \lim_n \int_E f_n$.

Proof:- since we have $|f_n| \leq g_n \; \forall n$ on E

$\Rightarrow -g_n \leq f_n \leq g_n$

$\Rightarrow 0 \leq -f_n + g_n \leq 2g_n$

Here $(g_n + f_n) \geq 0 \; \forall \; n$ on E

i.e., $\{g_n + f_n\}$ is a sequence of non-negative measurable function on E and $g_n + f_n \rightarrow g + f$ almost everywhere on E.

since by data, we have $g_n \rightarrow g$ almost everywhere on E

similarly $g_n - f_n \rightarrow g - f$ almost everywhere on E

Then by fatous lemma, we have

$$\int_E g + f \leq \underline{\lim_n} \int_E (g_n + f_n) = \underline{\lim_n} \int_E g_n + \underline{\lim_n} \int_E f_n$$

[By data we have $|f_n| \leq g_n \; \forall \; n$ on E. 'g' is integrable, so, f_n is also integrable, Thus $g_n + f_n$ and $g_n - f_n$ are also integrable]

$$\text{i.e., } \int_E g + \int_E f \leq \underline{\lim_n} \int_E g_n + \underline{\lim_n} \int_E f_n$$

$$= \int_E g + \int_E f_n$$

$$\text{i.e., } \int_E g + \int_E f \leq \int_E g + \underline{\lim_n} \int_E f_n$$

$$\Rightarrow \int_E f \leq \underline{\lim_n} \int_E f_n \quad \text{--- } (2)$$

since by (1), we have $g_n - f_n \geq 0 \; \forall \; n$ on E.

i.e., $\{g_n - f_n\}$ is a sequence of non-negative measurable functions and $g_n - f_n \rightarrow g - f$ almost everywhere on E

Then by fatous lemma, we have

$$\int_E g - f \leq \underline{\lim_n} \int_E g_n - f_n$$

$$= \underline{\lim_n} \int_E g_n + \underline{\lim_n} \int -f_n$$

$$= \underline{\lim_n} \int_E g_n - \overline{\lim_n} \int_E f_n$$

$$= \int_E g - \overline{\lim_n} \int_E f_n$$

i.e., $\int_E g - \int_E f \leq \int_E g - \overline{\lim_n} \int_E f_n$

$$\Rightarrow -\int_E f \leq -\overline{\lim_n} \int_E f_n \quad \text{—} (3)$$

From (2) and (3), we have

$$\overline{\lim_n} \int_E f_n \leq \int_E f \leq \underline{\lim_n} \int_E f_n \quad \text{—} (4)$$

But we know that $\overline{\lim_n} \int_E f_n \geq \underline{\lim_n} \int_E f_n$ — (5)

From (4) and (5), we have

$$\overline{\lim_n} \int_E f_n = \underline{\lim_n} \int_E f_n = \int_E f$$

i.e., $\int_E f = \lim_n \int_E f_n$

Hence proved.

Sec:- Convergence In Measure

Definition:- A sequence $\{f_n\}$ of measurable functions is said to be convergence to 'f' in measure if given $\epsilon > 0$ $\exists$ an integer $N \ni m\{x \,|\, |f_n(x) - f(x)| \geq \epsilon\} < \epsilon$ $\forall\, n \geq N$

Note:- $f_n \to f$ almost everywhere $\Rightarrow f_n \to f$ in measure.

Proof:- Suppose that $f_n \to f$ almost everywhere

Then by definition $\exists$ a measurable set $A \subset E$ with $m(A) = 0 \ni \lim_n f_n(x) = f(x)$ $\forall\, x \notin A$

We have to prove that $f_n \to f$ in measure.

In a contraction way suppose that $f_n \not\to f$ in measure. Then by definition, for a given $\epsilon > 0$ $\exists$ a

+ve integer $N \ni \{x \mid |f_n(x) - f(x)| \geq \epsilon\} > \epsilon \ \forall n \geq N$ —— (1)

since we have $m(A) = 0 \Rightarrow 0 = m(A)$

$\Rightarrow 0 = m\{x \mid |f_n(x) - f(x)| \geq \epsilon\}$

$\Rightarrow 0 \geq \epsilon$

which is a contradiction

Hence our supposition is wrong

$\therefore f_n \to f$ in measure

Hence proved.

Proposition 18:- let $\{f_n\}$ be a sequence of measurable functions that converges in measure to 'f' then there is a sub-sequence $\{f_{n_k}\} \to f$ almost everywhere.

Proof:- Given that $\{f_n\}$ is a sequence of measurable function that converges in measure.

Then by definition, for a given $\epsilon = 2^{-\nu} > 0$ $(\nu = 1, 2, 3 \ldots) \exists$ an integer $N \ni m\{x \mid |f_n(x) - f(x)| \geq 2^{-\nu}\} < 2^{-\nu} \ \forall n \geq N$ —— (1)

we have to prove that there is a sub-sequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to f almost everywhere on E.

i.e., By definition, it is sufficient to prove that there is a measurable set $A \subset E$ with $m(A) = 0$ such that $f_{n_k} \to f$ for each $x \notin A$

let $E_\nu = \{x \mid |f_n(x) - f(x)| \geq 2^{-\nu}\}$

Then $m(E_\nu) < 2^{-\nu} \ \forall \nu$ —— (2) [$\because$ from (1)]

consider $x \notin \bigcup_{\nu=k}^{\infty} E_\nu$

$\Rightarrow x \notin E_\nu$ for all $\nu \geq k$

$\Rightarrow |f_{n_\nu}(x) - f(x)| \geq 2^{-\nu} \ \forall \nu \geq k$

$\Rightarrow f_{n_q}(x) \to f(x) \; \forall \; x \notin \bigcup_{q=k}^{\infty} E_q$ —③

let $A = \bigcap_{k=1}^{\infty} \left[\bigcup_{q=k}^{\infty} E_q \right]$

Now if $x \notin A$ then $x \notin \bigcap_{k=1}^{\infty} \left[\bigcup_{q=k}^{\infty} E_q \right]$

$\therefore$ By eqn ③ $f_{n_q}(x) \to f(x) \; \forall \; x \notin A$

Now $m(A) = m\left[\bigcap_{k=1}^{\infty} \left[\bigcup_{q=k}^{\infty} E_q \right]\right]$

$$\leq \sum_{q=k}^{\infty} m(E_q) \; \forall \; k$$

$$\leq \sum_{q=k}^{\infty} 2^{-q}$$

$$= 2^{-k+1} \; \forall \; k$$

$\therefore m(A) < 2^{-k+1} \to 0$ as $k \to \infty$

$\therefore m(A) < 0$

But we have $m(A) \geq 0$

$\therefore m(A) = 0$

Hence $m(A) = 0 \ni f_{n_q}(x) \to f(x) \; \forall \; x \notin A$

Hence proved.

Problems:-

1. show that 'f' is integrable over E iff $|f|$ is integrable over E.

suppose 'f' is integrable over E

Then by definition, f^+, f^- are integrable over E

i.e., $\int_E f^+ < \infty$, $\int_E f^- < \infty$

we have to prove that $|f|$ is integrable over E

i.e., $\int_E |f| < \infty$

we know that $|f| = f^+ + f^-$

$\Rightarrow \int_E |f| = \int_E (f^+ + f^-)$

$= \int_E f^+ + \int_E f^-$

$\therefore \int_E |f| < \infty$

$\therefore |f|$ is integrable over E

Conversely, suppose that $|f|$ is integrable over E i.e,

$\int_E |f| < \infty$

we know that $f \le |f|$

$\Rightarrow \int_E f \le \int_E |f| < \infty$

$\Rightarrow \int_E f < \infty$

$\therefore$ 'f' is integrable over E

2. If 'f' and 'g' are measurable functions $\ni |f| \le |g|$ and if 'g' is integrable over E then show that f is integrable over E.

Given that 'f' and 'g' are measurable functions $\ni |f| \le |g|$ and 'g' is integrable over E.

Now we prove that 'f' is integrable over E

Now $|f| \le |g|$

$\Rightarrow f^+ + f^- \le |g|$

$\Rightarrow f^+ \le |g|$ and $f^- \le |g|$

$\Rightarrow \int_E f^+ < \infty$ and $\int_E f^- \le \int_E |g|$

$\Rightarrow \int_E f^+ < \infty$ and $\int_E f^- < \infty$

i.e, Both f^+, f^- are integrable over E

$\therefore$ 'f' is integrable over E

we know that $f = f^+ - f^-$

$\Rightarrow \int_E f = \int_E f^+ - \int_E f^-$

$< \infty$

$\therefore$ 'f' is integrable over E.

3. If 'f' is integrable over E then show that
$\left|\int_E f\right| \leq \int_E |f|$

Given that 'f' is integrable over E

Then by problem ① $|f|$ is also integrable over E

we know that $f \leq |f|$ and $-f \leq |f|$

$\Rightarrow f \leq |f|$ and $-|f| \leq f$

$\Rightarrow \int_E f \leq \int_E |f|$ and $\int_E -|f| \leq \int_E f$

$\Rightarrow \int_E f \leq \int_E |f|$ and $-\int_E |f| \leq \int_E f$

$-\int_E |f| \leq \int_E f \leq \int_E |f|$

$\Rightarrow \left|\int_E f\right| \leq \int_E |f|$ $\left[\because \int_E |f| \leq \int_E -|f| \right.$ &
$\left. -\int_E |f| \leq \int_E |f|\right]$

4. let 'f' be a non-negative measurable function
then $\int f = 0 \Rightarrow f = 0$ almost everywhere.

Given that 'f' be a non-negative measurable function.

Suppose $\int f = 0$ —①

claim:- $f = 0$ almost everywhere

let $A = \{x \in E \mid f(x) \neq 0\}$

i.e, $A = \{x \in E \mid f(x) > 0\}$ —Ⓐ

Now we have to show that $m(A) = 0$

let $A_n = \{x \in A \mid f(x) > \frac{1}{n}\}$ —②

since we have $\frac{1}{n} < f(x)$ on $A_n \Rightarrow \int_{A_n} \frac{1}{n} < \int_{A_n} f(x)$

$< \int_E f$ [By ①]

$\Rightarrow \int_{A_n} \frac{1}{n} = 0.$

$\Rightarrow \frac{1}{n}\int_{A_n}(1) = 0$

$\Rightarrow \frac{1}{n}\int \chi_{A_n}(x) \leq 0$

$\Rightarrow \frac{1}{n} m(A_n) \leq 0$

$\Rightarrow m(A_n) \leq 0$

But always we have $m(A_n) \geq 0$

$\therefore m(A_n) = 0 \ \forall n$ —③

Now $m(A) = m\left[\bigcup_n A_n\right]$ $\left[\text{By ② clearly } A = \bigcup_n A_n\right]$

$= \sum_n m(A_n)$

$\therefore m(A) = 0$

5. show that monotone convergence theorem need not hold for decreasing sequence of functions.

let us define $f_n(x)$ as follows

$$\left.\begin{aligned} f_n(x) &= 0 \text{ if } x < n \\ &1 \text{ if } x \geq n \end{aligned}\right\} \text{—①}$$

we have $f_1(x) = 0$ if $x < 1$
$= 1$ if $x \geq 1$

$f_2(x) = 0$ if $x < 2$
$= 1$ if $x \geq 2$

$f_3(x) = 0$ if $x < 3$
$= 1$ if $x \geq 3$

then $f_n(x) \geq f_{n+1}(x) \ \forall n$

so $\{f_n\}$ is decreasing sequence of measurable functions.

claim:– $\int_E f \neq \lim_n \int f_n$

Now for any $n > x$ by ① we have $f_n(x) = 0$

$\Rightarrow \lim_n f_n(x) = 0$ —②

put $f(x) = \underset{n}{\mathrm{lt}}\, f_n(x)\ \forall\ x$

(2) $\Rightarrow f(x) = 0\ \forall\ x$

$$\Rightarrow \int_E f = 0 \quad \text{--- (3)}$$

Now consider, $\int_{[-\infty,\infty]} f_n = \int_{[-\infty,n]} f_n + \int_{[n,\infty]} f_n$

$$= 0 + \infty = \infty$$

$$\Rightarrow \int_{[-\infty,\infty]} f_n = \infty$$

$$\underset{n}{\mathrm{lt}} \int f_n = \infty \quad \text{--- (4)}$$

From (3) and (4), we have $\int_E f \neq \underset{n}{\mathrm{lt}} \int f_n$

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